DOCUMENT RESUME

ED 135 835 TM 006 060

AUTHOR Echternacht, Gary
TITLE The Choice of Covariates.
PUB DATE [Apr 76]
NOTE 38p.; Paper presented at the Annual Meeting of the American Educational Research Association (60th, San Francisco, California, April 19-23, 1976)

EDRS PRICE MF-$0.83 HC-$2.06 Plus Postage.
DESCRIPTORS *Analysis of Covariance; Correlation; Error Patterns; *Observation; Probability; Program Evaluation; Statistical Bias

ABSTRACT

The analysis of covariance is frequently used in observational studies to either adjust treatment comparisons for biases due to different distributions of extraneous variables or for improving the precison of the treatment comparisons. This paper discusses aspects that the researcher should keep in mind when planning such a study. In particular it discusses which variables should be selected as covariates when there is a positive correlation among all the variables. (Author)

****
* Documents acquired by ERIC include many informal unpublished *
* materials not available from other sources. ERIC makes every effort *
* to obtain the best copy available. Nevertheless, items of marginal *
* reproducibility are often encountered and this affects the quality *
* of the microfiche and hardcopy reproductions ERIC makes available *
* via the ERIC Document Reproduction Service (EDRS). EDRS is not *
* responsible for the quality of the original document. Reproductions *
* supplied by EDRS are the best that can be made from the original. *
****

ED135835

The Choice of Covariates

Gary Echternacht
Educational Testing Service

Abstract

The analysis of covariance is frequently used in observational studies to either adjust treatment comparisons for biases due to different distributions of extraneous variables or for improving the precision of the treatment comparisons. This paper discusses aspects that the researcher should keep in mind when planning such a study. In particular it discusses which variables should be selected as covariates when there is a positive correlation among all the variables.

TM006 060

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
NATIONAL INSTITUTE OF EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

ED135835

TM006 060

The Choice of Covariates

Gary Echternacht
Educational Testing Service

Looking at the history of educational evaluation, one soon comes to the conclusion that the most popular evaluation design falls into a category that Campbell and Stanley (1963) term "quasi-experimental" designs, or as termed by Wold (1956), "observational studies." However one refers to such designs, their principal characteristic is that there is uncontrolled variation that has not been subjected to randomization.

The non-equivalent control group model has been used often, especially in evaluating large educational programs that are aimed at special groups with special needs. Typically whole classrooms or even schools are assigned to either an experimental treatment or control group. Rarely is this assignment random. Those students or schools judged "most in need" are assigned to the group receiving the experimental treatment, instruction, or services. This decision is made on moral grounds. If the experimental treatment is effective (almost all program administrators believe this), then those who are most in need should benefit from being in the experiment. This rationale is prevalent for programs that attempt to raise the achievement levels of low achieving students.

In evaluating programs that are designed to improve student achievement levels, a common design is to administer a standardized or objectives referenced test to both the treatment and control groups at the beginning and end of the program. A background questionnaire may be administered to all the students in the experiment to gain information on each student's socio-economic status

and other factors that may be related to the test scores. The analysis of data would aim to determine the effect or impact of the experimental program either by constructing confidence limits or testing statistical hypotheses about model parameters.

Any number of approaches might be used to analyze the data resulting from such an evaluation. Although some reject the notion out of hand (e.g., see Kenny, 1975), pair matching and stratification are sometimes used as a means of removing the biasing effects measured on the first testing from the treatment-control comparisons. Most frequently, however, data from such a design are analyzed using an analysis of covariance model, often using an approach that adjusts for measurement error in the pretest.

The analysis of covariance model is used in such instances for two reasons. First, it can be used to eliminate any bias in the comparison of the treatment group with the control group. Essentially this bias results from an uneven distribution of the covariate in the two groups. Second, it can be used to increase the precision of the estimate of the treatment effect. For example, if the covariate were scores on a standardized test, the score distribution for the control group might be a normal distribution with a mean of 50 and a standard deviation of 10. Scores for the treatment group might have a distribution with a negative skew with mean 40 and a standard deviation of 8. Comparison of the two groups at the second testing would be affected by pretest standings. Using the analysis of covariance would enable one to adjust the posttest means to remove the biasing effects of the difference in the pretest distributions.

One of the most common misconceptions about the analysis of covariance is that it is only used to adjust for these differences in the covariate distribution. Even though the covariate distributions are identical for the two groups, the analysis of covariance can be used to increase the precision of treatment-control comparisons (see Andersen and Bancroft, 1952, for example).

This paper deals with a problem that confronts everyone who plans an observational study and contemplates using the analysis of covariance to determine treatment effects. Many covariates will be available from background questionnaires and test scores. The question then becomes, what criteria can be established for selecting covariates?

McKinley (1975) notes that there is little in the existing literature on the selection of covariates, especially when multiple covariates are considered that are positively intercorrelated. Bross (1966 and 1967) provides a nice rule of thumb and methodology for selecting a single covariate or estimating the likely effect of including another variable in the analysis. Unfortunately, Bross's methodology applies only to dichotomously coded variables and are not very useful in an educational evaluation context. Cochran (1965), on the other hand, provides a useful methodology for continuous variables but gives no guidelines for selecting covariates from among those that are highly correlated. This paper attempts to supply such guidelines though they are limited to the case of two covariates and can be used only with large samples where degrees of freedom are not a concern.

Policy during planning

When planning an observational study, Cochran (1965) suggests constructing a list of variables that is subsequently arranged in three groups. The first group consists of major variables that should be included as covariates. This list should be small in view of the complexities in adjusting for many variables simultaneously. The second group consists of variables that we would like to think have little or no effect on the analysis. The third list contains those variables that are thought to be minor and will be discarded.

The most important important decision during the planning of an observational study is when to include a variable in the first list and when to include a variable in the second list. The most reasonable policy is to include every possible covariate on the first list, which will guide test selection and questionnaire development. This recommendation is based on the notion that the loss incurred by asking for information and later not using that information is small compared with the loss incurred by not asking for the information and later realizing that it may be important. Multiple computer runs of the analysis of covariance are relatively inexpensive and allow the analyist to examine alternate models for the data.

Perhaps the above should be tempered somewhat. When selecting covariates there are two things that should be kept in mind. We want to protect against bias entering into the estimate of treatment effect. We also want to keep in mind that even if there seems to be no danger of any bias entering our estimate, the presence of a disturbing

variable may inflate the variance of the effect to such an extent that comparisons become imprecise.

Selecting covariates to adjust for bias

Suppose we consider the case where we have two groups, with each group having a continuous variable y designated as a dependent variable. We assume a model with covariates x_1 and x_2 that takes the form,

$$y = \mu_y + \phi(x_1,x_2) + e$$

in the treatment group and in the control group,

$$y = \mu_{y'} + \phi(x_1',x_2') + e'$$

where the residuals e and e' have zero means. The regression of y on x_1 and x_2 may take any form, but must be expressed as a continuous function and must be the same in both populations. In most instances we would be interested in estimating the treatment effect $\mu_y - \mu_{y'}$.

Using the above, we can write

$$\bar{y} - \bar{y}' = \mu_y - \mu_{y'} + \bar{\phi} - \bar{\phi}' + \bar{e} - \bar{e}'.$$

If $\bar{y} - \bar{y}'$ is to be an unbiased estimate of $\mu_y - \mu_{y'}$, then $E(\bar{\phi} - \bar{\phi}')$ must be zero. Since the regression of y on x_1 and x_2 takes a continuous form, it is possible to use Taylor's expansion to write,

$$\phi(x_1,x_2) = \beta_{00} + \beta_{10}x_1 + \beta_{01}x_2 + \beta_{20}x_1^2 + \beta_{11}x_1x_2 + \beta_{02}x_2^2 + \ldots$$

so that the bias due to x_1 and x_2 in $\bar{y} - \bar{y}'$ is

$$\beta_{10}(\mu_{10} - \mu_{10}') + \beta_{01}(\mu_{01} - \mu_{01}') + \beta_{20}(\mu_{20} - \mu_{20}') + \beta_{11}(\mu_{11} - \mu_{11}') + \beta_{02}(\mu_{02} - \mu_{02}') + \ldots\ldots$$

where $\mu_{ij} = E(x_1^i,x_2^j)$ and $\mu_{ij}' = E(x_1'^i,x_2'^j)$ are the i,j th moments about zero in the two populations. Verification that the sample

means $\bar{x}_1$, $\bar{x}_1'$ and $\bar{x}_2$, $\bar{x}_2'$ do not differ by more than sampling error gives assurance that bias from a linear regression of y on x_1 and x_2 is small. A check that the two samples have the same means, variances, and covariances among the covariates is assurance against a quadratic regression. By looking at higher order moments, one can similarly eliminate biases due to higher order regression.

Examining statistics for bias

After the data has been collected, one usually decides to use a variable as a covariate on the basis of a significance test between the treatment and control group means. Cochran (1965) points out that it is not clear what kind of assurance is given in finding a non-significant result. Nor is it appropriate, he says, to always use a t-test as the criterion. Cochran gives the following illustration for a case with one covariate, where the regression of y on x is known to be linear.

In repeated samples with $(\bar{x} - \bar{x}')$ fixed, the distribution of $(\bar{y} - \bar{y}')$ is normal with mean $\mu - \mu' + \beta(\bar{x} - \bar{x}')$ and variance $2\sigma_e^2/n$. If there is no bias due to x, $\bar{y} - \bar{y}'$ is normally distributed with mean $\mu - \mu'$ and variance $2(\sigma_e^2 + \beta^2\sigma_x^2)/n$. A 95 percent confidence interval for $\mu - \mu'$ can then be calculated by

$$\bar{y} - \bar{y}' \pm 1.96\sqrt{(2(\sigma_e^2 + \beta^2\sigma_x^2)/n)}$$

From the conditional distribution of $\bar{y} - \bar{y}'$ given above, the probability that the above limits include $\mu - \mu'$ is the probability that a normal deviate lies between the limits

$$\sqrt{(n/2)}\ \beta(\bar{x} - \bar{x}')/\sigma_e \pm 1.96\sqrt{(1 + \beta^2\sigma_x^2/\sigma_e^2)}$$

Using the test criterion

$$t = \sqrt{(n/2)}\ (\bar{x} - \bar{x}')/\ \sigma_x$$

and letting $v = \beta\sigma_x/\sigma_e$, we have the probability that the calculated limits include $\mu - \mu'$ being the probability that a normal deviate lies between

$$-tv \pm 1.96(1 + v^2)$$

Cochran develops a table, which is reproduced as Table 1, showing the conditional probabilities that the 95 percent confidence limits calculated from above actually include $\mu - \mu'$ for different values of v and t.

Table 1

Probability that the 95 percent limits include $\mu - \mu'$

	v				
t	0.3	0.4	0.5	0.6	1.0
0.5	.957	.961	.966	.972	.988
1.0	.950	.950	.951	.953	.963
1.5	.948	.931	.924	.917	.898
2.0	.922	.903	.882	.862	.785

If t<1, the table indicates the probabilities are greater than 95 percent. If t = 1.5, the probabilities are about correct, especially when v is small. For t = 2, the probabilities are unsatisfactory even when v is small.

Cochran suggests that one should take a second look at those covariates with t-values of about 1.5 after the values of y become known. An estimate of v can be made and the unadjusted $\bar{y} - \bar{y}'$ compared with the adjusted estimate to see if there is any material

difference. If several covariates have t-values substantially above 1.5, Cochran suggests that the investigator consider whether the two groups are suitable for comparison.

Using covariates to increase precision: one covariate

The most simple case involving the selection of a covariate requires us to choose between the following two models

$$\text{Model A:} \quad \begin{aligned} y &= \mu_y + \beta x + e \text{ (treatment group)} \\ y &= \mu_y, + \beta x' + e' \text{ (control group)} \end{aligned}$$

and

$$\text{Model B:} \quad \begin{aligned} y &= \mu_y + e^* \text{ (treatment group)} \\ y &= \mu_y + e^{*\prime} \text{ (control group)} \end{aligned}$$

We will assume that $E(x) = E(x')$ so that the primary reason for including the covariate is to increase the precision of the comparison of the treatment group mean with the control group mean. Using model A, we have seen that the variance of $\bar{y} - \bar{y}'$ is given by $2(\sigma_e^2 + \beta^2\sigma_x^2)/n$. In model B the variance of $\bar{y} - \bar{y}'$ is given by $2\sigma_{e^*}^2/n$. If there is a linear regression of y on x in both populations, β^2 can be written as $\rho_{xy}^2\sigma_y^2/\sigma_x^2$. Substituting this into the formula for the variance of y using model A and solving for the error term, one finds that

$$\sigma_e^2 = (1 - \rho_{xy}^2)\sigma_y^2$$

In model B $\sigma_y^2 = \sigma_{e^*}^2$, or σ_y^2 represents the error variance that would occur without a covariance adjustment. The increase in precision occurring in model A is due to the correlation between the dependent variable and the covariate. This increase in

precision can be expressed in terms of the percent reduction in the error variance when the covariate is added to the model. Table 2 gives some reductions in the error for different population correlation coefficients.

Table 2
Reduction in error variance

ρ_{xy}	Percent reduction in error variance
.10	1.0%
.20	4.0%
.25	6.25%
.30	9.0%
.35	12.25%
.40	16.0%
.50	25.0%
.60	36.0%
.70	49.0%

The above has several implications for the design of observational studies. For example, if the dependent variable and covariate are parallel forms of a standardized achievement test, they will have a high correlation. It is easy to see that a pretest-posttest design will be superior to a posttest only design even when randomization is used to assign students to treatment and control groups. On the other hand, if only background variables that correlate low with student achievement are used as a covariate one could not expect a dramatic reduction in the error.

Using covariates to increase precision: two correlated covariates

In this case we make the assumption that x_1 has already been selected as a covariate, and we wish to judge the merits of adding x_2 as a second covariate. In other words, we wish to choose between the models

$$\text{Model A:} \quad \begin{aligned} y &= \mu_y + \beta_1 x_1 + \beta_2 x_2 + e \\ y' &= \mu_{y'} + \beta_1 x_1' + \beta_2 x_2' + e' \end{aligned}$$

and

$$\text{Model B:} \quad \begin{aligned} y &= \mu_y + \beta x_1 + e^* \\ y' &= \mu_y + \beta x_1' + e^{*\prime} \end{aligned}$$

In model A the population coefficients β_1 and β_2 can be expressed in the form

$$\beta_1 = \frac{\sigma_y(\rho_{x_1y} - \rho_{x_1x_2}\rho_{x_2y})}{\sigma_{x_1}(1 - \rho^2_{x_1x_2})}$$

$$\beta_2 = \frac{\sigma_y(\rho_{x_2y} - \rho_{x_1x_2}\rho_{x_1y})}{\sigma_{x_2}(1 - \rho^2_{x_1x_2})}$$

If these expressions are substituted for β_1 and β_2 in the appropriate equation for σ^2_e, and the equation is subsequently solved for σ^2_e in terms of the remaining variables, σ^2_e becomes

$$\sigma^2_e = \frac{\sigma^2_y}{(1 - \rho^2_{x_1x_2})}(1 - \rho^2_{x_1y} - \rho^2_{x_2y} - \rho^2_{x_1x_2} + 2\rho_{x_1y}\rho_{x_2y}\rho_{x_1x_2})$$

In model B, $\sigma^2_{e^*}$ becomes

$$\sigma^2_{e^*} = \sigma^2_y(1 - \rho^2_{x_1y})$$

Assuming that each of the correlations is positive, σ_e^2 will be less than σ_{e*}^2. Sometimes the difference will be extremely small and will not justify prefering the more complex model A over model B. At other times the difference between σ_e^2 and σ_{e*}^2 will be substantial, and the precision of the treatment comparison will be greatly improved by using model A. The percentage reduction in error resulting from the use of model A can be obtained by dividing σ_e^2 by σ_{e*}^2, subtracting the result from one, and multiplying the result by 100 percent. Percentage reductions in error are shown in Table 3 for various combinations of ρ_{x_1y}, ρ_{x_2y}, and $\rho_{x_1x_2}$.

Perhaps the most important thing to notice when adding a second covariate is the way in which $\rho_{x_1x_2}$ and ρ_{x_2y} affect the resulting improvement in precision. The relationship between the percent error reduction and $\rho_{x_1x_2}$ and ρ_{x_2y} is shown in Figures 1 to 11. In Figures 1 to 5 the percent reduction in error is graphed as a function of $\rho_{x_1x_2}$ for ρ_{x_1y} = .3, .4, .5, .6, and .7. Each curve represents the effect of the correlation between covariates for different combinations of ρ_{x_1y} and ρ_{x_2y}. The figures show that the general effect of the correlation between covariates is to reduce the effectiveness of the second covariate as a means for increasing precision. Nevertheless, there is a point at which an increasing correlation between covariates no longer reduces the effectiveness of the second covariate. The percentage reduction in error begins to increase with $\rho_{x_1x_2}$. This becomes most pronounced when ρ_{x_1y} is high.

For fixed values of ρ_{x_1y} and ρ_{x_2y} it is possible to find the

value of $\rho_{x_1x_2}$ where the percentage error reduction has reached a minimum. This turns out to occur at the point

$$\frac{\min\{\rho_{x_1y}, \rho_{x_2y}\}}{\max\{\rho_{x_1y}, \rho_{x_2y}\}}$$

where min designates the smaller of the two values and max designates the larger of the two values.

In examining Figure 5 --a case where we have already selected a very useful covariate-- we see that the initial fall-off in the percent error reduction due to an increasing correlation between covariates is extremely steep. On the one hand, if we were to choose a second covariate that correlates highly with the dependent variable, our effort may be for naught if the second covariate correlates about as high with the first covariate as it does with the dependent variable. On the other hand, we might be better off looking for a second covariate that correlates low with x_1 and more modestly with y. If we were extremely lucky, we might find a variable that low with y and yet improves precision because of its high correlation with x_1. Such a variable would be one that accounted for initial differences at the beginning of the treatment and subsided by the end of the treatment. An example of this might be found in the comparison of a group of Spanish speaking students who speak little English with a control group of students who speak English sufficiently well. If a pretest-posttest design is used and the treatment is an English reading program, at pretesting the ability

to speak English would be highly correlated with a pretest given in English. As time passed, however, the Spanish speaking children might have learned to speak English much better as a result of associating with English speaking children. By the time of posttesting, the Spanish speaking children may speak English about as well as those children in the control group. Such an effect is closely related to what Campbell and Stanley (1963) call the interaction between selection and maturation. In this example, the second covariate is attempting to account for that interaction.

Figures 6 to 11 show the percentage error reduction as a function of ρ_{x_2y} for fixed levels of ρ_{x_1y} and $\rho_{x_1x_2}$. These functions are also quadratics with a point where the percentage reduction in error reaches a minimum. For fixed values of ρ_{x_1y} and $\rho_{x_1x_2}$ the minimum percent reduction in error occurs when the correlation between the second covariate and the dependent variable is $\rho_{x_1y}\rho_{x_1x_2}$.

In looking through the the figures, two general rules of thumb stand out. First, if $\rho_{x_2y}>.3$, adding that variable as the second covariate is always worthwhile as long as $\rho_{x_1x_2}<.2$. If $\rho_{x_2y}>\rho_{x_1y}$, there appears to be a worthwhile reduction in the percent error reduction regardless of the value of $\rho_{x_1x_2}$.

Recommendations for planning observational studies

All of the above discussion is concerned with the planning of observational studies that plan to use the analysis of covariance to either reduce bias or improve precision. All of the discussion has been in terms of population statistics that are assumed to be known during the planning. There has been no mention of how sample sizes affect the percent reductions in error. Therefore, the

recommendations given must be tempered somewhat.

What does all of this mean to the researcher planning to use the analysis of covariance with observational data? First, it suggests that the researcher should ask for as much data as possible that might be used for covariates. Ideally these should not correlate highly with each other unless one is worried about a possible interaction between the groups and maturation. In such a case the researcher should collect data relating to that interaction. The model he specifies should include such factors if warranted by the data. Second, after the covariate data is collected, the researcher should look at the means, standard deviations, and covariances of the covariates for the treatment and control groups. Verification that these are the same for for each group will insure against bias entering into the treatment comparison, at least up to a quadratic relationship among the covariates. If, however, the groups differ substantially on many covariates, the researcher must decide whether it is possible to compare the groups meaningfully. Third, the researcher might use Figures 1 to 11 to judiciously select covariates from a larger pool to improve the precision of the treatment comparison. Fourth, regardless of how many or which covariates are selected, the various adjusted and unadjusted estimates should be compared. Multiple analyses using different combinations of covariates are encouraged.

References

Anderson, R. L. and Bancroft, T. A. Statistical Theory in Research. New York: McGraw-Hill, 1952

Bross, D. J. Spurious effects from an extraneous variable. Journal of Chronic Diseases, 1966, 19, 637 - 647.

Bross, D. J. Pertinency of an extraneous variable. Journal of Chronic Diseases, 1967, 20, 487-495.

Campbell, D. T. and Stanley, J. C. Experimental and quasi-experimental designs for research on teaching. In N. L. Gage(Ed.) Handbook of Research on Teaching. Chicago: Rand McNally, 1963.

Cochran, W. G. The planning of observational studies of human populations. Journal of the Royal Statistical Society, Series A, part 2, 1965, 234-266.

Kenny, D. A. A quasi-experimental approach to assessing treatment effects in the non-equivalent control group design. Psychological Bulletin, 82, 345-362.

McKinley, S. M. The design and analysis of the observational study--a review. Journal of the American Statistical Association, 1975, 70, 503-520.

Wold, H. Causal inference from observational data. Journal of the Royal Statistical Society, Series A, part 1, 1956, 28-60

Table 3

Percent reductions in error

ρ_{x_1y}	ρ_{x_2y}	$\rho_{x_1x_2}$ 0.0	.2	.3	.4	.5	.6
.3	.2	4.40	2.24	1.46	.84	.37	.07
	.3	9.89	6.59	5.33	4.24	3.30	2.47
	.4	17.58	13.23	11.61	10.26	9.16	8.31
	.5	27.47	22.16	20.30	18.89	17.95	17.58
	.6	39.56	33.38	31.41	30.14	29.67	30.29
.4	.2	4.76	1.79	.84	.23	.00	.30
	.3	10.71	6.00	4.24	2.78	1.59	.67
	.4	19.05	12.70	10.26	8.16	6.35	4.76
	.5	29.76	21.88	18.89	16.38	14.28	12.57
	.6	42.86	33.53	30.27	27.43	25.40	24.11
.5	.2	5.33	1.39	.37	.00	.44	2.08
	.3	12.00	5.56	3.30	1.59	.44	.00
	.4	21.33	12.50	9.16	6.35	4.00	2.08
	.5	33.33	22.22	17.95	14.29	11.11	8.33
	.6	48.00	34.72	29.67	25.40	21.78	18.75
.6	.2	6.25	1.04	.07	.30	2.08	6.25
	.3	14.06	5.27	2.47	.67	.00	.88
	.4	25.00	12.76	8.31	4.76	2.08	.39
	.5	39.06	23.50	17.58	12.57	8.33	4.79
	.6	56.25	37.50	30.29	24.11	18.75	14.06
.7	.2	7.84	.74	.02	1.49	5.88	14.83
	.3	17.65	5.23	1.75	.09	.65	4.41
	.4	31.37	13.81	7.78	3.36	.65	.12
	.5	49.02	26.47	18.12	11.30	5.88	1.96
	.6	70.59	43.22	32.77	23.90	16.34	9.93

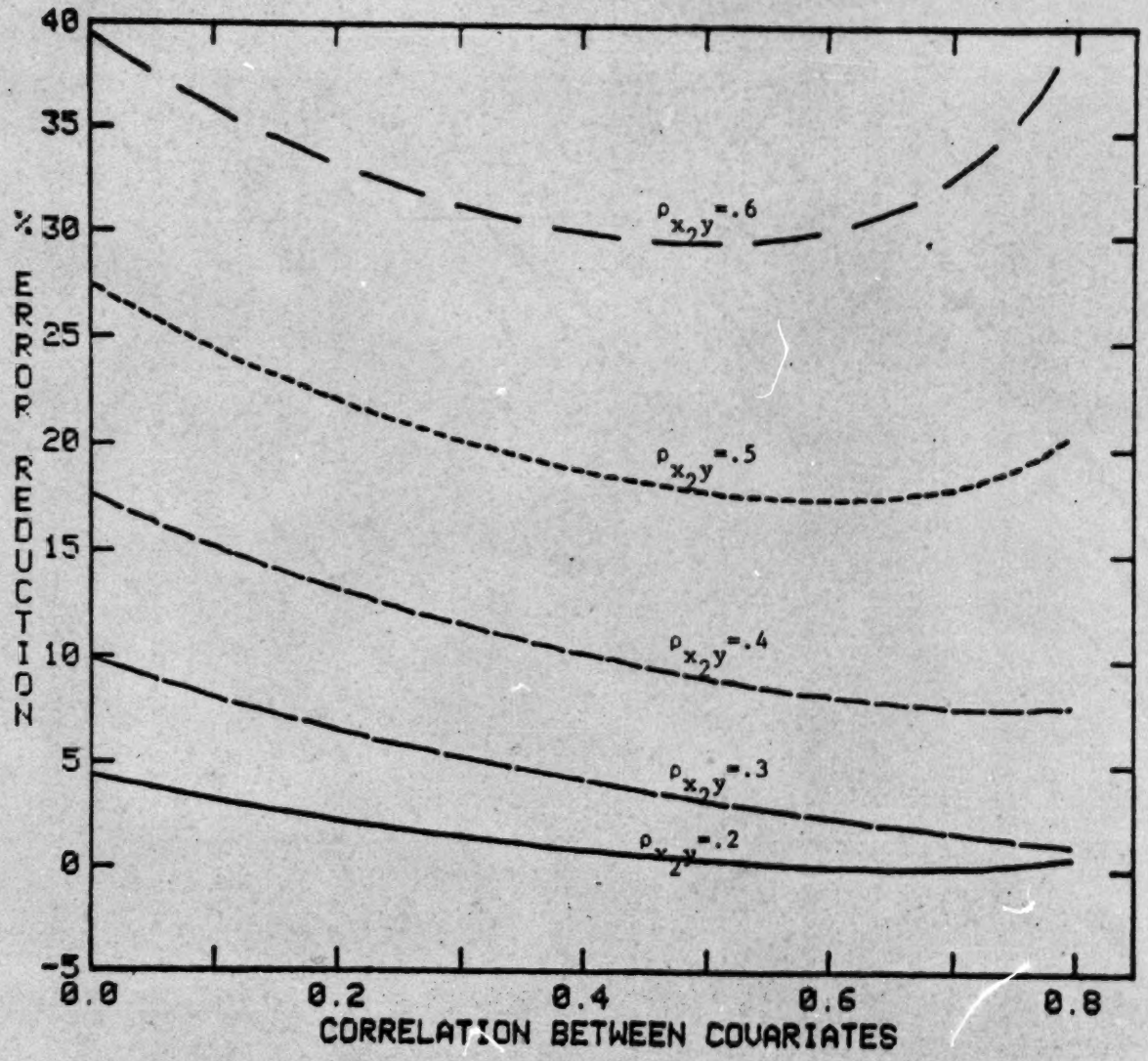
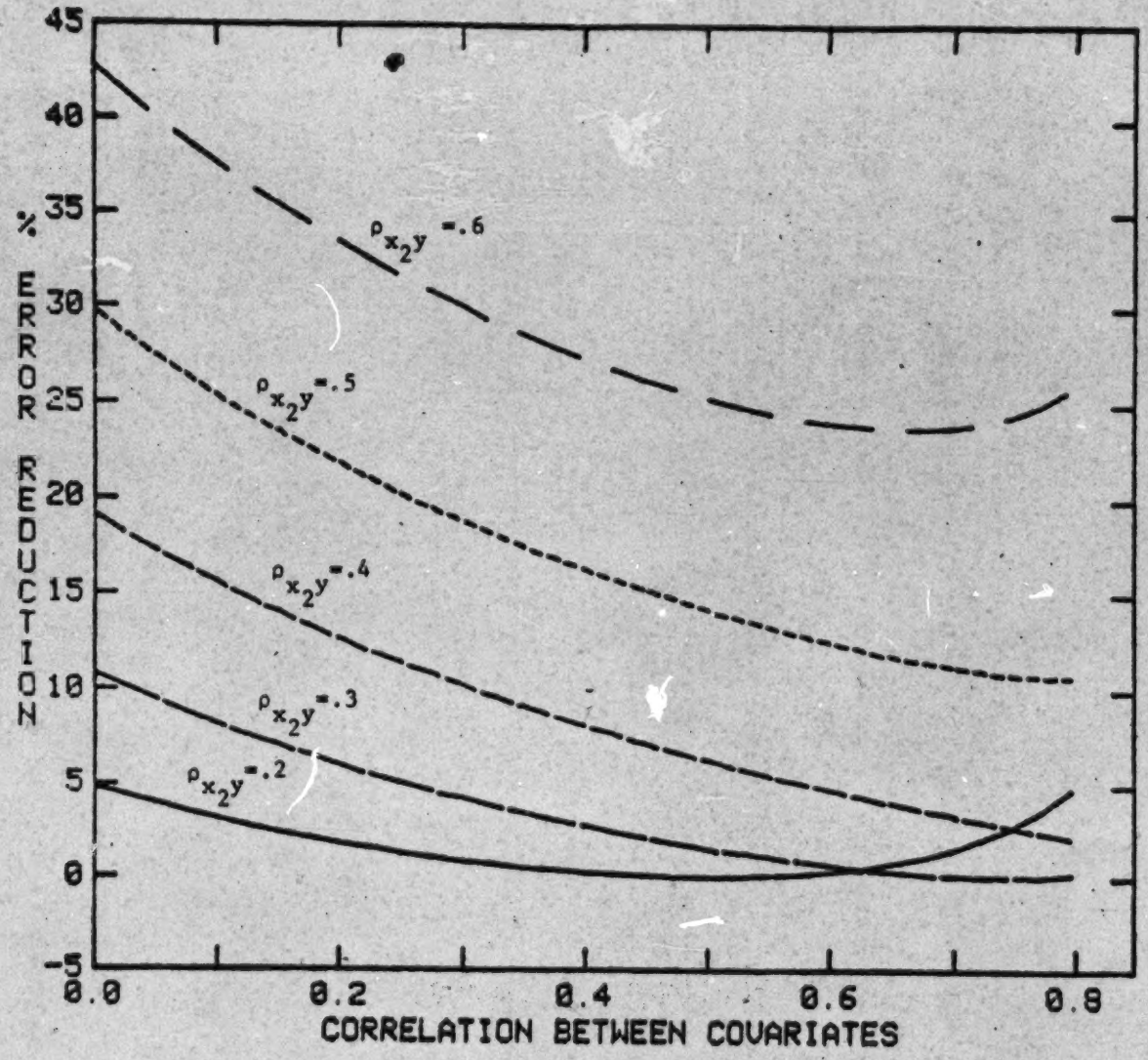


Figure 1. Percent error reduction as a function of $\rho_{x_1x_2}$ for ρ_{x_1y} = .3



21

Figure 2.. Percent error reduction as a function of $\rho_{x_1x_2}$ for ρ_{x_1y} = .4

22

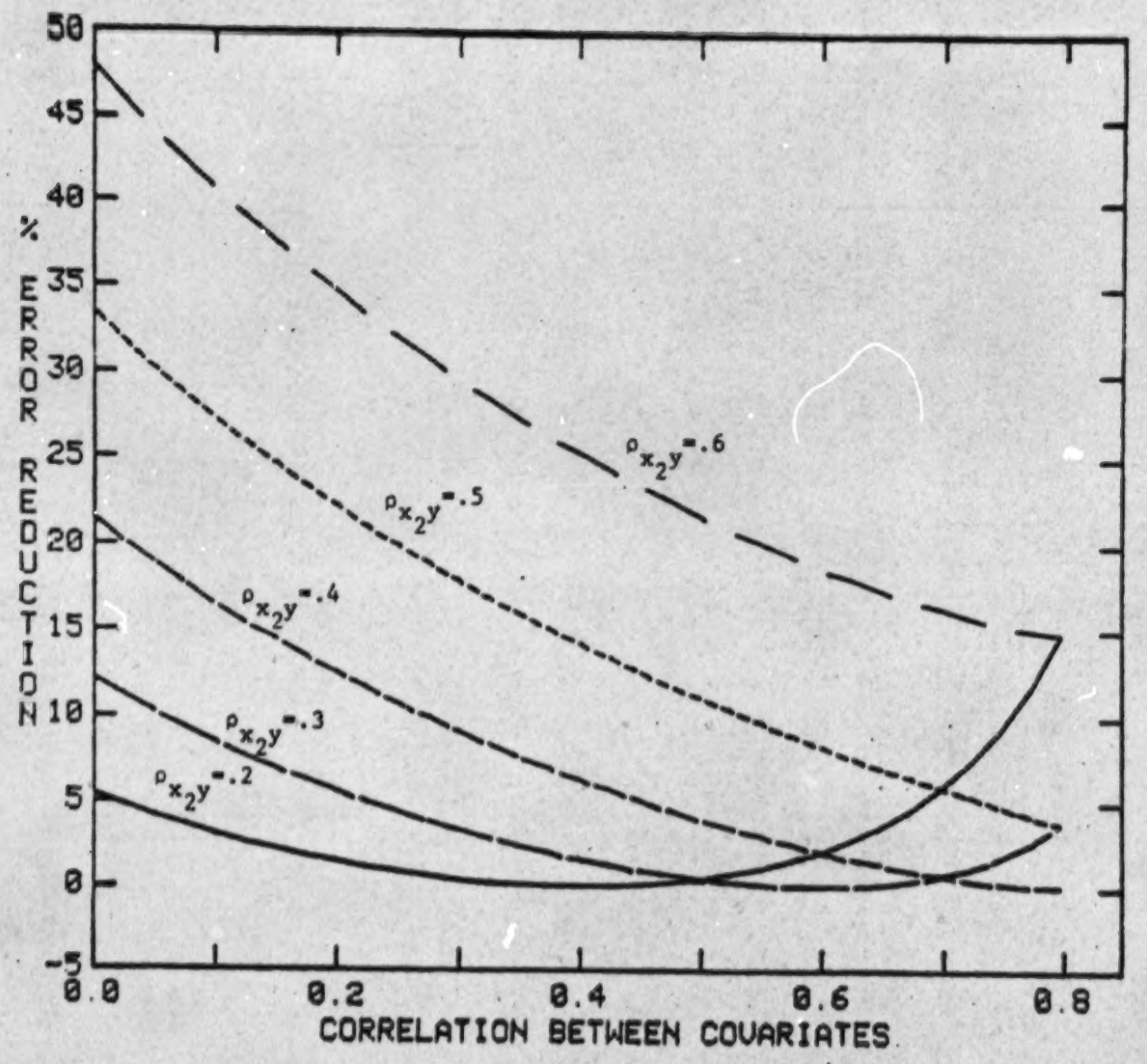


Figure 3. Percent error reduction as a function of $\rho_{x_1x_2}$ for ρ_{x_1y}=.5

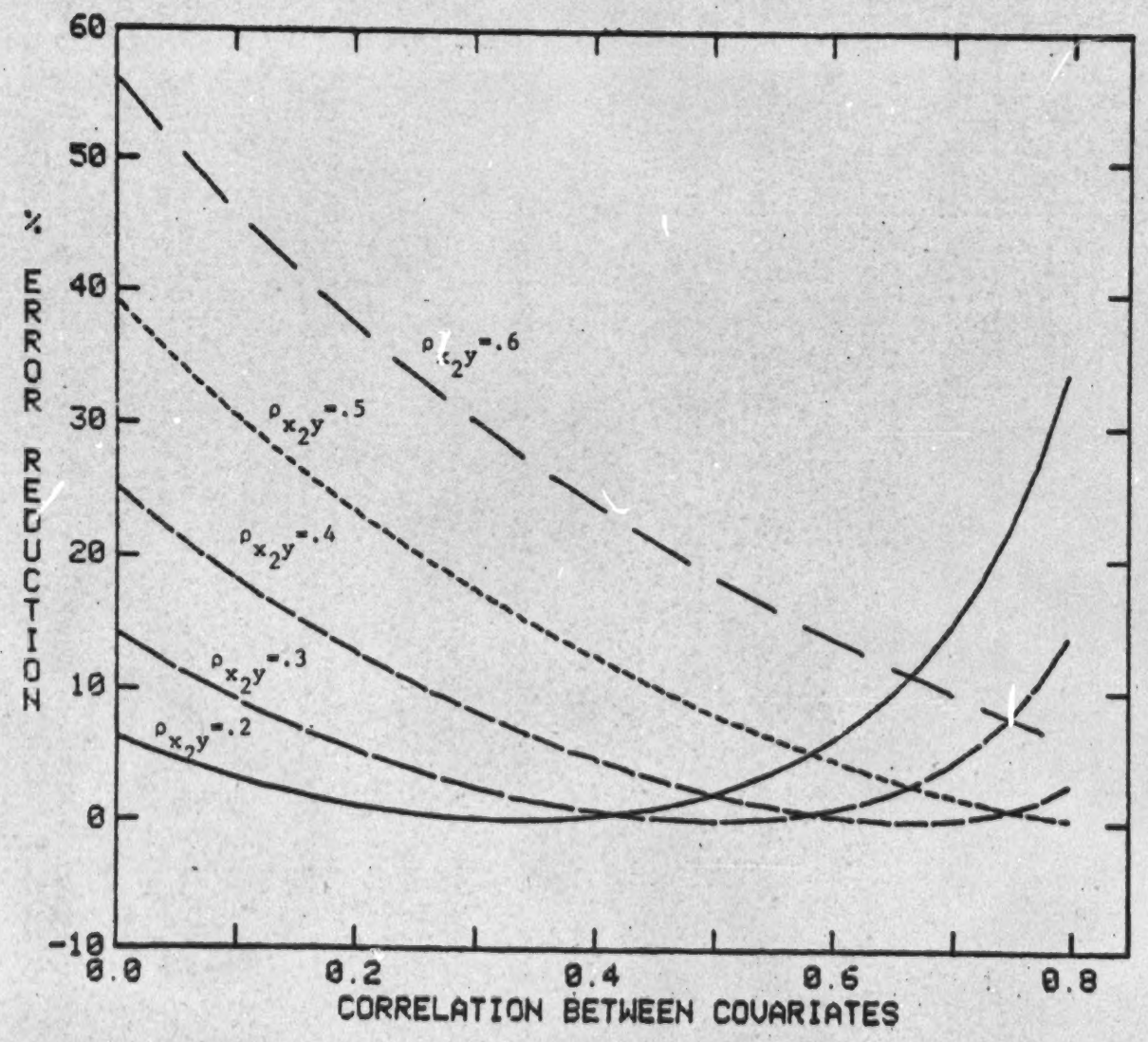


Figure 4. Percent error reduction as a function of $\rho_{x_1x_2}$ for ρ_{x_1y}=.6

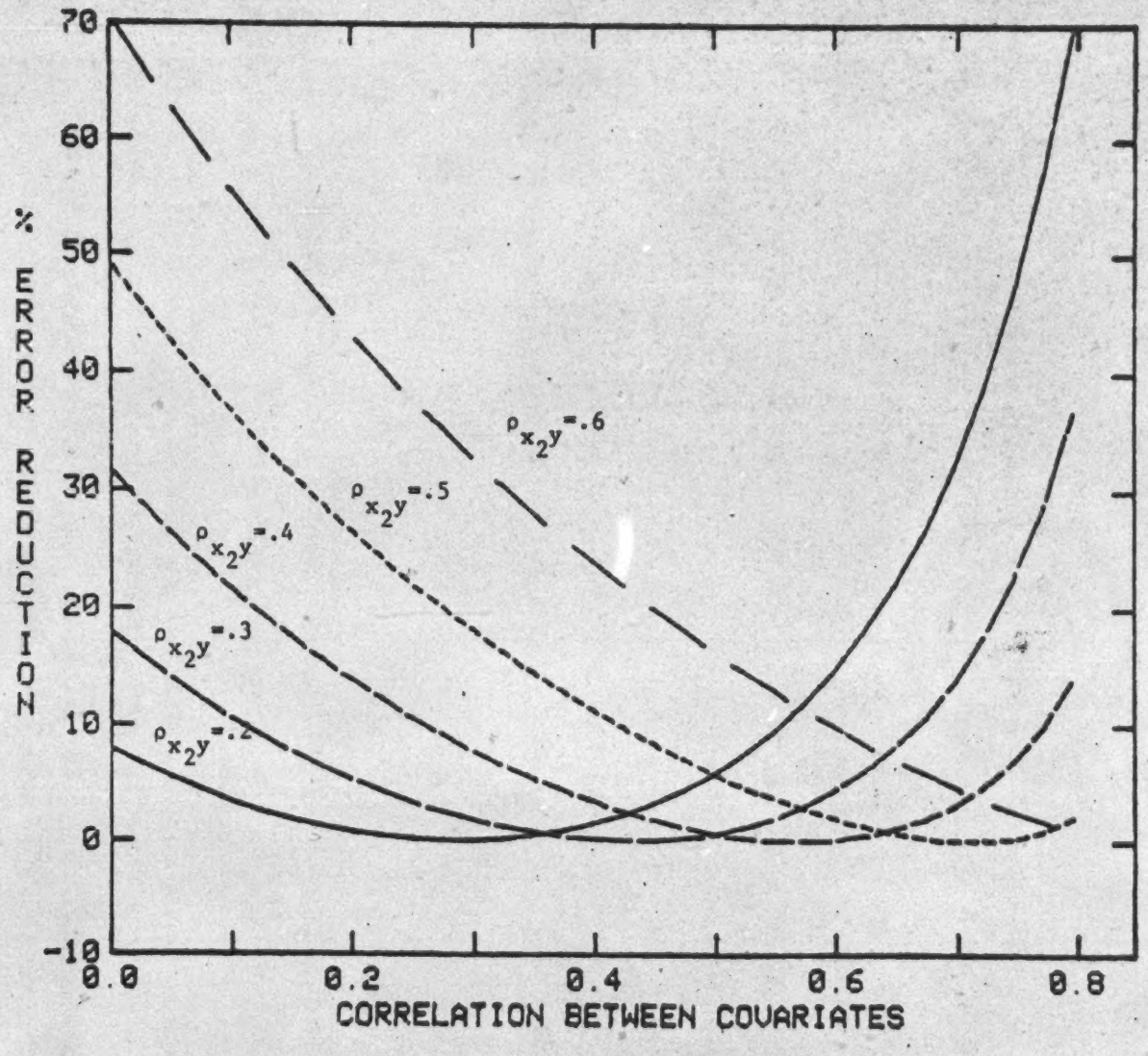


Figure 5. Percent error reduction as a function of $\rho_{x_1x_2}$ for ρ_{x_1y}=.7

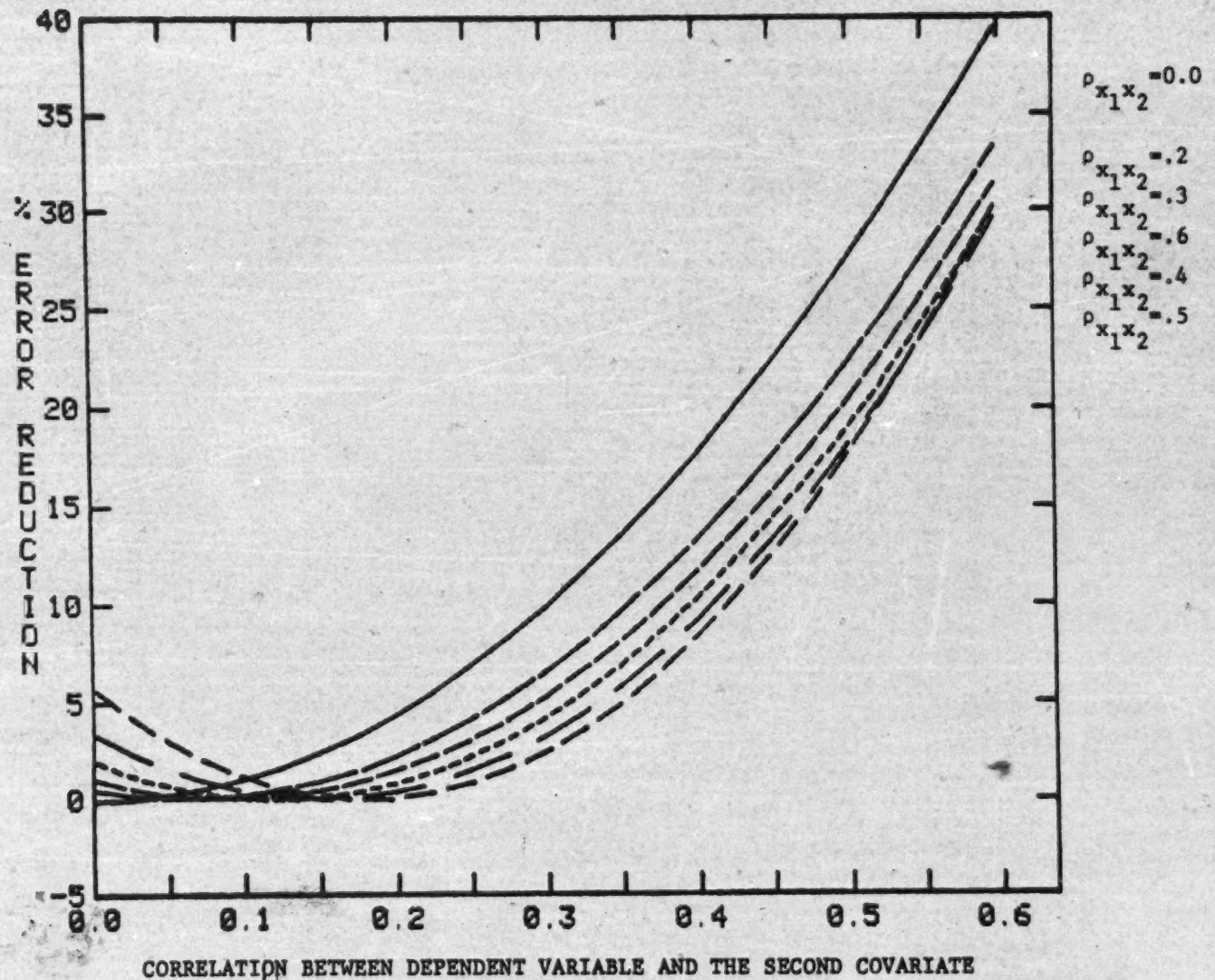


Figure 6 Percent error reduction as a function of $\rho_{x_{2y}}$ for $\rho_{x_{1y}}$=.3

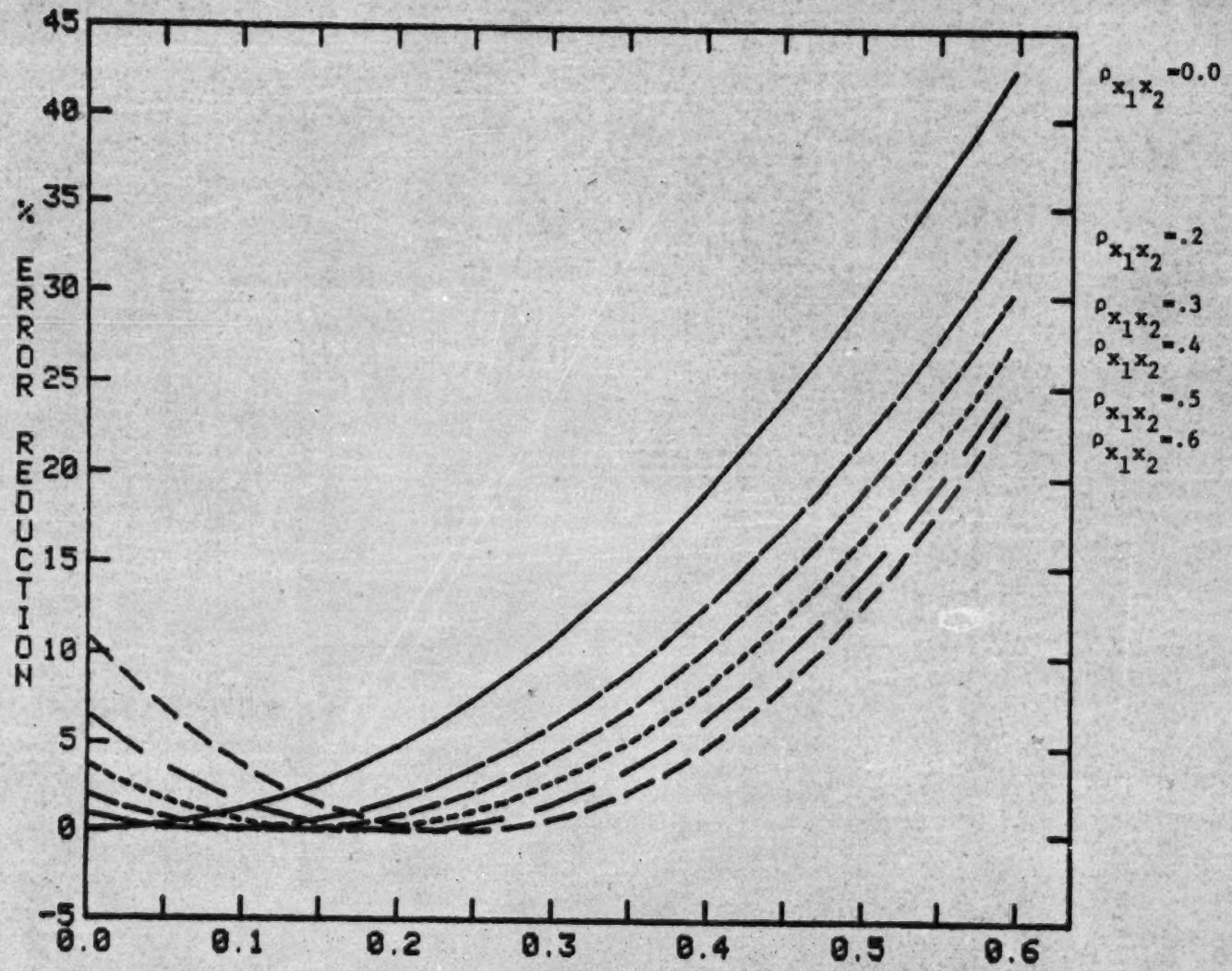


Figure 7. Percent error reduction as a function of ρ_{x_2y} for ρ_{x_1y}=.4

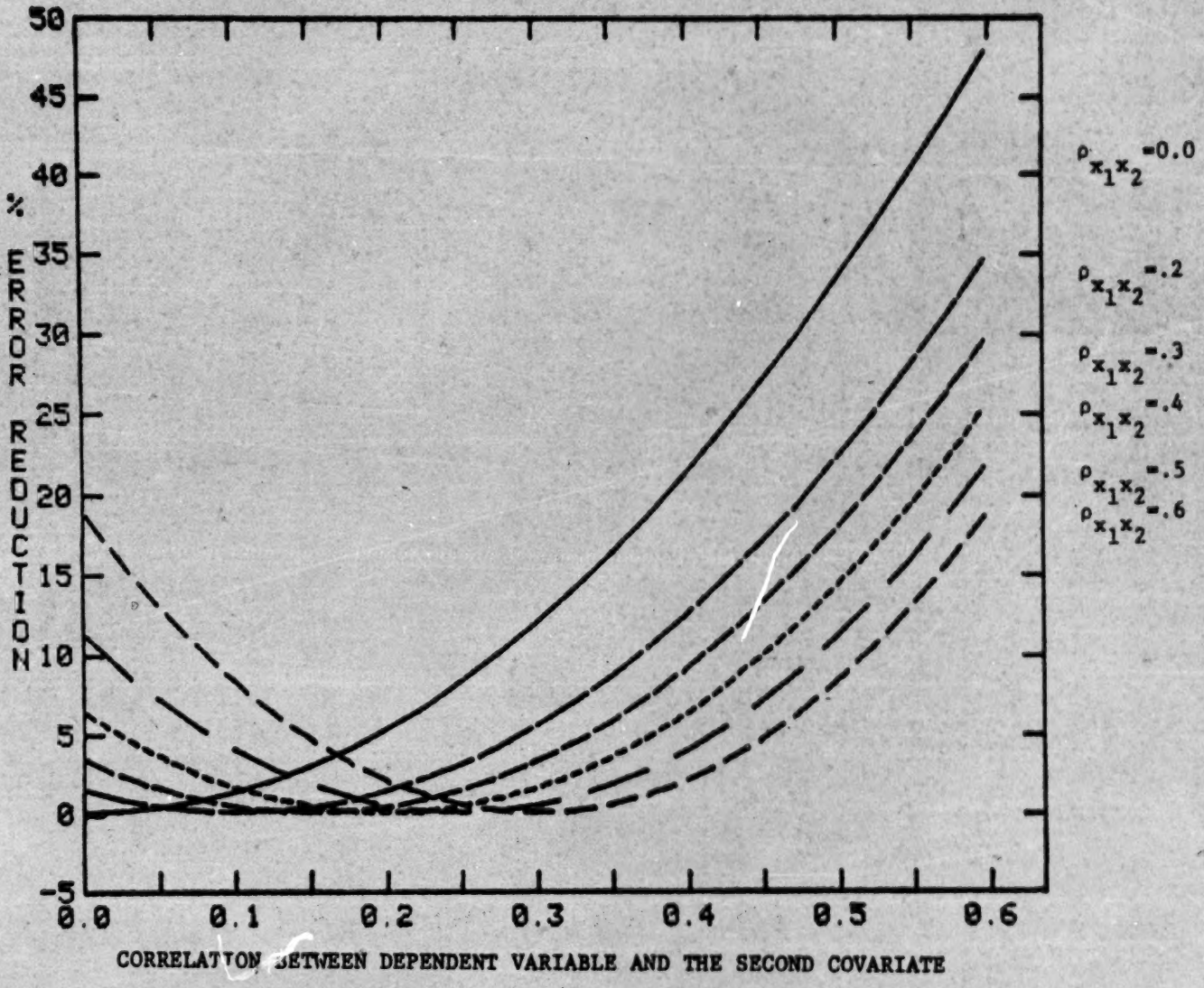


Figure 8 Percent reduction in error as a function of $\rho_{x_{2y}}$ for ρ_{x_1y}=.5

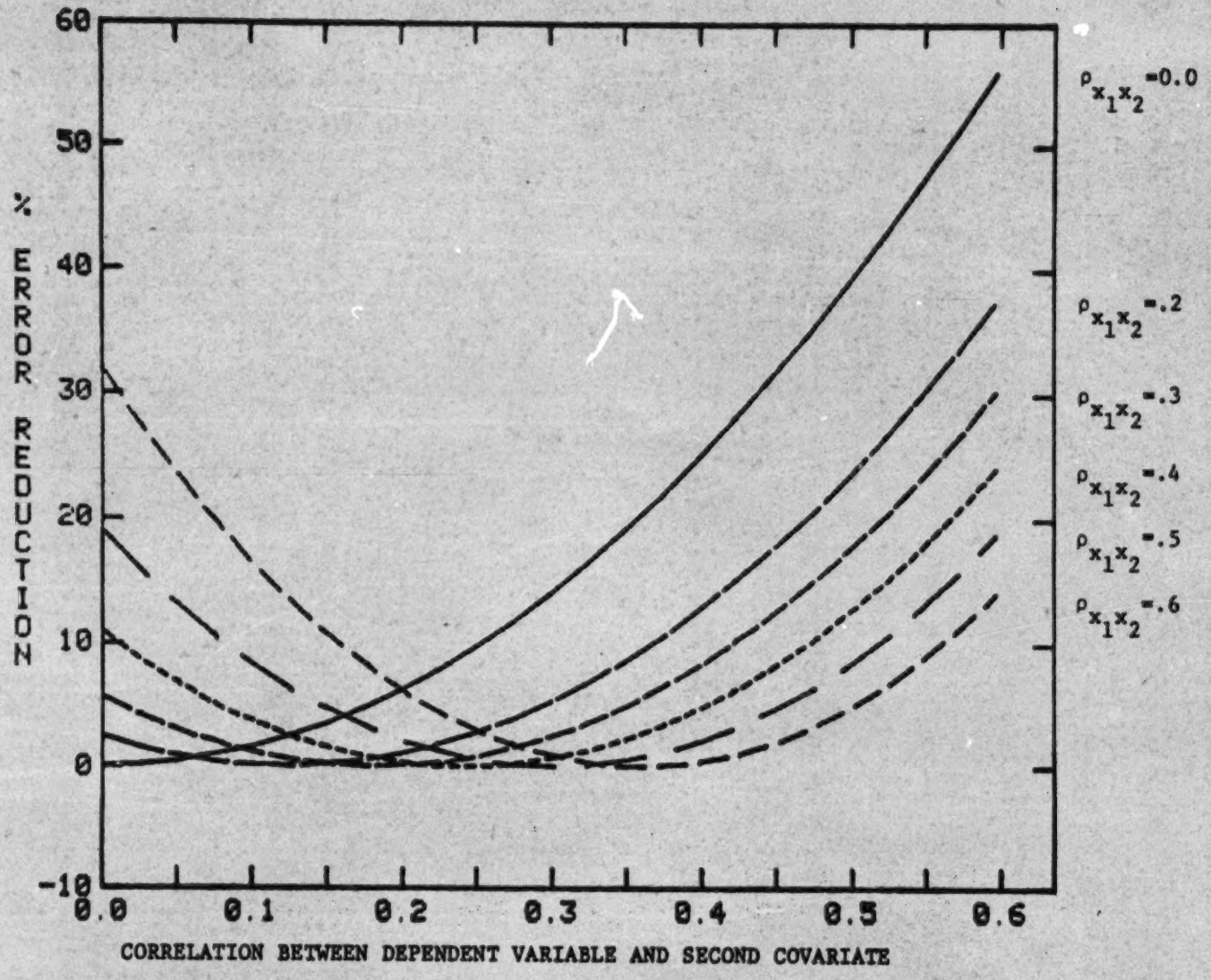


Figure 9. Percent error reduction as a function of ρ_{x_2y} for ρ_{x_1y}=.6

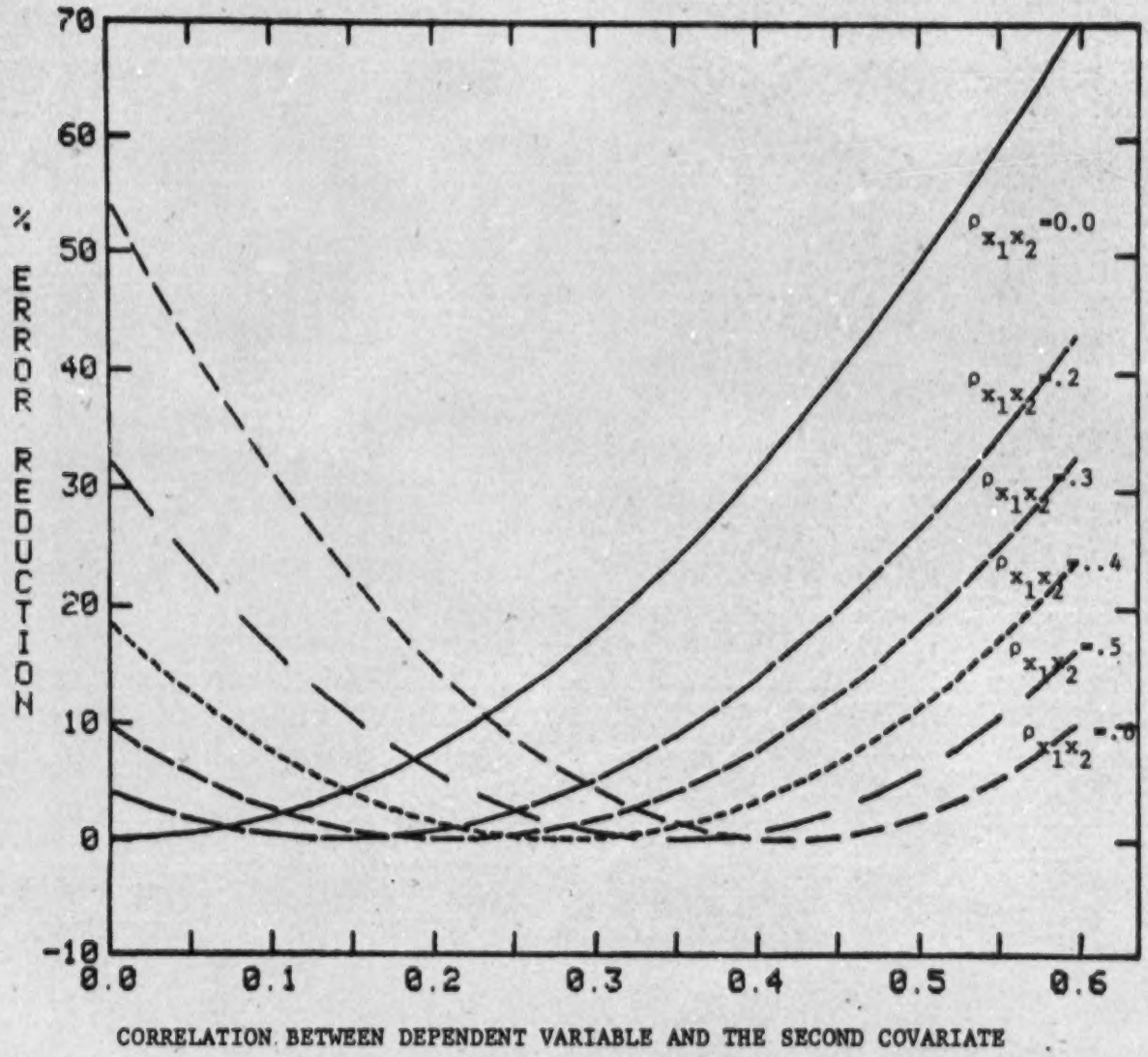


Figure 10. Percent error reduction as a function of ρ_{x_2y} for ρ_{x_1y}=.7